

PERVERSE SHEAVES AND QUANTUM GROTHENDIECK RINGS

M. VARAGNOLO AND E. VASSEROT

ABSTRACT. We define a quantum analogue of the Grothendieck ring of finite dimensional modules of a quantum affine algebra of simply laced type via an analogue of Lusztig's restriction functor on perverse sheaves on a variety related to quivers. We get also a new geometric construction of the tensor category of finite dimensional modules of a finite dimensional simple Lie algebra of type $A - D - E$.

1. INTRODUCTION

Finite-dimensional representations of quantum affine algebras, say $\mathbf{U} = \mathbf{U}_q(\mathbf{Lg})$, have been studied from various viewpoints. However, little is known on the decomposition factors of tensor product of simple modules. From Lusztig's work the direct sum of the Grothendieck rings of affine Hecke algebras of type A can be identified with the algebra of regular functions of the pro-unipotent group of upper triangular unipotent $\mathbb{Z} \times \mathbb{Z}$ -matrices with finite support, in such a way that simple modules are mapped to the dual canonical basis of $\mathbf{U}_q^+(\mathfrak{sl}_\infty)$. It was observe recently that the induction product of simple modules of affine Hecke algebra should be related to conjectural multiplicative properties of the dual canonical basis, see [NLT]. The aim of our paper is to give a similar approach for tensor product of simple modules for all simply laced types, using the geometric realization of quantum affine algebras in [N2], see also [GV], [Va] for type A . In order to do this we give a geometric construction of a flat deformation, denoted by \mathbf{GR} , of the Grothendieck ring of \mathbf{U} in terms of perverse sheaves on a singular variety related to quivers. The product is defined via an analogue of Lusztig's restriction functor. It is not commutative in general, and \mathbf{GR} affords a canonical basis. Note that \mathbf{GR} and its canonical basis appeared already in [N3] in a different form. It was also observed, there, that the elements of the canonical basis could be identified to simple \mathbf{U} -modules with a prescribed, conjectural, filtration. However, the construction in [N3] does not give the positivity statement in Theorem 4.3. There is no geometric construction of the tensor category of finite-dimensional \mathbf{U} -modules. The positivity in Theorem 4.3 suggests that a large number of information on tensor products of \mathbf{U} -modules can be captured from the ring \mathbf{GR} . In particular, we formulate a generalization of a conjecture of Berenstein-Zelevinsky, see [BZ].

A similar construction gives a new geometric interpretation of the tensor category of finite dimensional \mathfrak{g} -modules, see §5. It would be interesting to relate it with

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the tensor category of perverse sheaves on the affine Grassmanian of the Langlands dual group. This question appeared independently in [M].

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2. THE GROTHENDIECK RINGS

2.1. Let \mathfrak{g} be a simple complex Lie algebra with Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let $d_i \in \{1, 2, 3\}$ be the coprime positive integers such that the matrix with entries $b_{ij} = d_i a_{ij}$ is symmetric. Let α_i and ω_i be the simple roots and the fundamental weights of \mathfrak{g} . Set $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i$ and let P^+, Q^+ be the semi-groups generated by $\{\alpha_i\}$ and $\{\omega_i\}$. Recall that Q is embedded in P by the linear map such that $\alpha_i \mapsto \sum_j a_{ji}\omega_j$. For any $\lambda \in P$, $\alpha, \beta \in Q$ we write $\beta \geq \alpha$ if $\beta - \alpha \in Q^+$ (resp. we write $\lambda \geq \alpha$ if $\lambda - \alpha \in P^+$). If $\lambda \in P^+$ let $V(\lambda)$ be the simple \mathfrak{g} -module with highest weight λ . If $\lambda \in P$ and V is an integrable \mathfrak{g} -module, let $V_\lambda \subseteq V$ be the corresponding weight subspace in V . We put

$$\Lambda(\lambda) = \{\alpha \in Q^+ \mid V(\lambda)_{\lambda-\alpha} \neq \{0\}\}, \quad \Lambda^+(\lambda) = \{\alpha \in \Lambda(\lambda) \mid \lambda \geq \alpha\}.$$

Let $\mathbf{R}(\mathfrak{g})$ be the ring of finite dimensional representations of \mathfrak{g} . In this paper, except in §5.2, we consider only simply laced Lie algebras.

2.2. The quantum loop algebra associated to \mathfrak{g} is the $\mathbb{C}(q)$ -algebra \mathbf{U} generated by \mathbf{x}_{ir}^\pm , $\mathbf{k}_{i,\pm s}^\pm$, $\mathbf{k}_i^{\pm 1} = \mathbf{k}_{i0}^\pm$ ($i \in I$, $r \in \mathbb{Z}$, $s \in \mathbb{N}$) modulo the following defining relations

$$\begin{aligned} \mathbf{k}_i \mathbf{k}_i^{-1} &= 1 = \mathbf{k}_i^{-1} \mathbf{k}_i, \quad [\mathbf{k}_{i,\pm r}^\pm, \mathbf{k}_{j,\varepsilon s}^\varepsilon] = 0, \\ \mathbf{k}_i \mathbf{x}_{jr}^\pm \mathbf{k}_i^{-1} &= q^{\pm a_{ij}} \mathbf{x}_{jr}^\pm \\ (z - q^{\pm a_{ji}} w) \mathbf{k}_j^\varepsilon(z) \mathbf{x}_i^\pm(w) &= (q^{\pm a_{ji}} z - w) \mathbf{x}_i^\pm(w) \mathbf{k}_j^\varepsilon(z) \\ (z - q^{\pm a_{ij}} w) \mathbf{x}_i^\pm(z) \mathbf{x}_j^\pm(w) &= (q^{\pm a_{ij}} z - w) \mathbf{x}_j^\pm(w) \mathbf{x}_i^\pm(z) \\ [\mathbf{x}_{ir}^+, \mathbf{x}_{js}^-] &= \delta_{ij} \frac{\mathbf{k}_{i,r+s}^+ - \mathbf{k}_{i,r+s}^-}{q - q^{-1}} \end{aligned}$$

$$\sum_w \sum_{p=0}^{1-a_{ij}} (-1)^p \begin{bmatrix} 1 - a_{ij} \\ p \end{bmatrix}_i \mathbf{x}_{ir_w(1)}^\pm \mathbf{x}_{ir_w(2)}^\pm \cdots \mathbf{x}_{ir_w(p)}^\pm \mathbf{x}_{js}^\pm \mathbf{x}_{ir_w(p+1)}^\pm \cdots \mathbf{x}_{ir_w(1-a_{ij})}^\pm = 0$$

where $i \neq j$, $r_1, \dots, r_{1-a_{ij}} \in \mathbb{Z}$ and $w \in S_{1-a_{ij}}$. Here we have set $\varepsilon = +$ or $-$,

$$[n] = (q^n - q^{-n})/(q - q^{-1}), \quad [n]! = [n][n-1]\cdots[2], \quad \begin{bmatrix} m \\ p \end{bmatrix} = \frac{[m]!}{[p]![m-p]!},$$

$$\mathbf{k}_i^\pm(z) = \sum_{r \geq 0} \mathbf{k}_{i,\pm r}^\pm z^{\mp r} \quad \text{and} \quad \mathbf{x}_i^\pm(z) = \sum_{r \in \mathbb{Z}} \mathbf{x}_{ir}^\pm z^{\mp r}.$$

Let $\mathbf{U}^\pm \subset \mathbf{U}$ be the subalgebra generated by the elements $\mathbf{x}_{i,r}^\pm$ with $i \in I$, $r \in \mathbb{Z}$. For a future use, we also introduce the elements $\mathbf{h}_{is} \in \mathbf{U}$, $s \neq 0$, such that

$$\mathbf{k}_i^\pm(z) = \mathbf{k}_i^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{s \geq 1} \mathbf{h}_{i,\pm s} z^{\mp s}\right).$$

Let Δ be the coproduct defined in terms of the Kac-Moody generators $\mathbf{e}_i, \mathbf{f}_i, \mathbf{k}_i^{\pm 1}$, $i \in I \cup \{0\}$, of \mathbf{U} as follows

$$\Delta(\mathbf{e}_i) = \mathbf{e}_i \otimes 1 + \mathbf{k}_i \otimes \mathbf{e}_i, \quad \Delta(\mathbf{f}_i) = \mathbf{f}_i \otimes \mathbf{k}_i^{-1} + 1 \otimes \mathbf{f}_i, \quad \Delta(\mathbf{k}_i) = \mathbf{k}_i \otimes \mathbf{k}_i.$$

2.3. Fix $\lambda = \sum_i \ell_i \omega_i \in P^+$, $\alpha = \sum_i a_i \alpha_i \in Q^+$, $G_\alpha = \prod_i \mathrm{GL}_{a_i}$, $G_\lambda = \prod_i \mathrm{GL}_{\ell_i}$. For any algebraic group G let G^\vee be the set of cocharacters of G , and let $G^{\vee, \mathrm{ad}}$ be the set of conjugacy classes in G^\vee . Thus $(\mathbb{C}^\times)^\vee = \{q^k; k \in \mathbb{Z}\}$. The direct sum $\mathrm{GL}_m^\vee \times \mathrm{GL}_n^\vee \rightarrow \mathrm{GL}_{m+n}^\vee$ gives a semigroup structure on the sets $X^+ = \bigsqcup_{\lambda \in P^+} G_\lambda^{\vee, \mathrm{ad}}$, $Y^+ = \bigsqcup_{\alpha \in Q^+} G_\alpha^{\vee, \mathrm{ad}}$. The Abelian groups X, Y associated to X^+, Y^+ , are identified with the groups $\mathbb{Z}[q^{-1}, q] \otimes_{\mathbb{Z}} P$, $\mathbb{Z}[q^{-1}, q] \otimes_{\mathbb{Z}} Q$ via the maps

$$\gamma \mapsto \sum_i (\mathrm{tr} \gamma_i) \otimes \omega_i, \quad \eta \mapsto \sum_i (\mathrm{tr} \eta_i) \otimes \alpha_i,$$

where γ_i, η_i are the i -th components of the elements $\gamma \in G_\lambda^{\vee, \mathrm{ad}}$, $\eta \in G_\alpha^{\vee, \mathrm{ad}}$. Hereafter we may omit the symbol \otimes and write simply $q^n \lambda$ instead of $q^n \otimes \lambda$. Consider the $\mathbb{Z}[q, q^{-1}]$ -linear map

$$\Omega : Y \rightarrow X, \quad \alpha_i \mapsto [2]\omega_i - \sum_{a_{ij}=-1} \omega_j.$$

Hereafter, let $\gamma + \eta$ denote the element $\gamma + \Omega(\eta) \in X$. We write $\eta \succeq \delta$ if $\eta, \delta \in Y$ are such that $\eta - \delta \in Y^+$ (resp. we write $\gamma \succeq \eta$ if $\gamma \in X$, $\eta \in Y$ are such that $\gamma - \eta \in X^+$). We have $q^{-1}\Omega = A + q^{-1}B + q^{-2}A$, where A, B are $\mathbb{Z}[q, q^{-1}]$ -linear operators such that $A(\alpha_i) = \omega_i$ for all $i \in I$. Let

$$(\mid) : (\mathbb{Z}((q^{-1})) \otimes_{\mathbb{Z}} Q) \times X \rightarrow \mathbb{Z}((q^{-1}))$$

be the $\mathbb{Z}((q^{-1}))$ -bilinear form such that $(\alpha_i \mid \omega_j) = \delta_{ij}$. Let $\Omega^{-1} : X \rightarrow \mathbb{Z}((q^{-1})) \otimes Q$ be the inverse of Ω . For any $\gamma, \gamma' \in X$, we put

$$\varepsilon_{\gamma\gamma'} = (q^{-1}\Omega^{-1}(\bar{\gamma}) \mid \gamma')_0, \quad \langle \gamma, \gamma' \rangle = \varepsilon_{\gamma\gamma'} - \varepsilon_{\gamma'\gamma},$$

where f_0 is the constant term of a formal series f , and $\bar{}$ is the \mathbb{Z} -linear involution such that $\bar{q} = q^{-1}$. It is easy to see that

$$\varepsilon_{\gamma+\gamma', \gamma''} + \varepsilon_{\gamma\gamma'} = \varepsilon_{\gamma, \gamma'+\gamma''} + \varepsilon_{\gamma'\gamma''},$$

for all $\gamma, \gamma', \gamma'' \in X^+$. Put $\mathbb{A} = \mathbb{Z}[v, v^{-1}]$. Let \mathbf{A}_X be the \mathbb{A} -algebra linearly spanned by elements e^γ , $\gamma \in X$, such that

$$(1) \quad e^\gamma \cdot e^{\gamma'} = v^{\langle \gamma, \gamma' \rangle} e^{\gamma+\gamma'}.$$

2.4. The simple finite dimensional \mathbf{U} -modules are labelled by I -uples of monic polynomials in $\mathbb{C}(q)[t]$ with nonzero constant terms, called the Drinfeld polynomials. If $\gamma = \sum_k \gamma_k \in X^+$ with $\gamma_k = q^{n_k} \omega_{i_k}$ and $n_k \in \mathbb{Z}$, let $V(\gamma)$ be the simple finite dimensional \mathbf{U} -module whose i -th Drinfeld polynomial is $P_\gamma^{(i)}(z) = \prod_{i_k=i} (z - q^{n_k})$. For any \mathbf{U} -module V and any I -uple of formal series $\psi^\pm = (\psi_i^\pm) \in \mathbb{C}(q)[[z^{\pm 1}]]^I$, set $V_\psi = \bigcup_N \bigcap_i \mathrm{Ker} (\mathbf{k}_i^\pm(z) - \psi_i^\pm \mathrm{Id})^N \subseteq V$. If $\gamma = \gamma^+ - \gamma^-$ with $\gamma^\pm \in X^+$, we put $V_\gamma = V_\psi$ where ψ_i^\pm is the expansion at ∞ or 0 of the rational function

$$q^{(\gamma(1) \mid \alpha_i)} \cdot P_\gamma^{(i)}(1/qz) \cdot P_\gamma^{(i)}(q/z)^{-1},$$

and $P_\gamma^{(i)} = P_{\gamma^+}^{(i)} / P_{\gamma^-}^{(i)}$. Let \mathcal{C}_q be the category of pairs (V, F) where V is a finite dimensional \mathbf{U} -module such that $V = \bigoplus_\gamma V_\gamma$, and F is a decreasing \mathbb{Z} -filtration on V compatible with the weight decomposition, i.e. $F_\ell V = \bigoplus_\gamma (V_\gamma \cap F_\ell V)$ for all ℓ . Let $\mathbf{K}(\mathcal{C}_q)$ be the \mathbb{A} -module with one generator for each $(V, F) \in \text{Ob}(\mathcal{C}_q)$ modulo the relations

$$- (V, F) = (V', F') + (V'', F'') \text{ if there is an exact sequence}$$

$$0 \rightarrow (V', F') \rightarrow (V, F) \rightarrow (V'', F'') \rightarrow 0,$$

$$- (V, F) = v(V', F') \text{ if } (V, F) \text{ is isomorphic to } (V', F'[1]).$$

Fix an element $\gamma = \sum_{k=1}^\ell \gamma_k \in X^+$, with $\gamma_k = q^{n_k} \omega_{i_k}$, such that $n_1 \geq n_2 \geq \dots \geq n_\ell$. The \mathbf{U} -module $V(\gamma_1) \otimes V(\gamma_2) \otimes \dots \otimes V(\gamma_\ell)$ does not depend on the choice of such a decomposition of γ , since

$$V(q^n \omega_i) \otimes V(q^n \omega_j) \simeq V(q^n \omega_j) \otimes V(q^n \omega_i)$$

for all n, i, j , see [Ka]. Let denote it by $W(\gamma)$. Fix a highest weight vector $v_\gamma \in V(\gamma)$. It is known that $W(\gamma)$ is a cyclic \mathbf{U}^- -module generated by the monomial $w_\gamma = v_{\gamma_1} \otimes v_{\gamma_2} \otimes \dots \otimes v_{\gamma_\ell}$, see [Ka], [VV]. The geometric construction in [N2] implies that

$$W(\gamma) = \bigoplus_{\gamma'} W(\gamma)_{\gamma'}, \quad \mathbf{x}_{ir}^-(W(\gamma)_{\gamma'}) \subseteq \bigoplus_{\gamma''} W(\gamma)_{\gamma''},$$

where the sum is over all elements $\gamma'' \in \gamma' - q^{\mathbb{Z}} \alpha_i$, see also [FM]. Note that the element \mathbf{x}_{ir}^- is not homogeneous for the weight decomposition above. Let $x_{ir}^{(t)}$ be its component in

$$\bigoplus_{\gamma'} \text{Hom}(W(\gamma)_{\gamma'}, W(\gamma)_{\gamma' - q^t \alpha_i}).$$

Set also

$$\phi_{ir}^{(t)} = \sum_{s=0}^r \binom{r}{s} (-1)^{r-s} q^{-st} x_{is}^{(t)}.$$

We endow $W(\gamma)$ with the decreasing \mathbb{Z} -filtration such that

$$\begin{aligned} - \{0\} &= F_1 W(\gamma)_\gamma \subset F_0 W(\gamma)_\gamma = W(\gamma)_\gamma, \\ - F_k W(\gamma)_{\gamma''} &= \sum_{i,r,t} \phi_{ir}^{(t)} (F_\ell W(\gamma)_{\gamma'}), \end{aligned}$$

where ℓ, γ' are such that

$$\gamma' = \gamma'' + q^t \alpha_i, \quad \ell = k - 2r - 1 - g''_{i,t+1} + g''_{i,t-1}, \quad \gamma'' = \sum_{i,k} g''_{ik} q^k \cdot \omega_i.$$

We have $(W(\gamma), F) \in \text{Ob}(\mathcal{C}_q)$. There is a unique surjective homomorphism of \mathbf{U} -modules $W(\gamma) \rightarrow V(\gamma)$ such that $w_\gamma \mapsto v_\gamma$. The module $V(\gamma)$ is endowed with the quotient filtration. Hereafter, the classes of the pairs $(V(\gamma), F)$, $(W(\gamma), F)$ in $\mathbf{K}(\mathcal{C}_q)$ are simply denoted by $V(\gamma)$, $W(\gamma)$. Let $\mathbf{GR} \subset \mathbf{K}(\mathcal{C}_q)$ be the \mathbb{A} -submodule spanned by the elements $V(\gamma)$. The tensor product of two objects in \mathcal{C}_q is endowed with the filtration such that

$$(1) \quad F_k(V_\gamma \otimes V_{\gamma'}) = \sum_{\ell + \ell' = k + \langle \gamma, \gamma' \rangle} F_\ell V_\gamma \otimes F_{\ell'} V_{\gamma'}.$$

Put

$$\text{gdm}(V_\gamma, F) = \sum_\ell \dim(Gr_\ell^F V_\gamma) \cdot v^\ell, \quad \text{gch}(V, F) = \sum_\gamma \text{gdm}(V_\gamma, F) \cdot e^\gamma,$$

where Gr^F is the associated graded space.

Proposition. (a) The map $\text{gch} : \mathbf{K}(\mathcal{C}_q) \rightarrow \mathbf{A}_X$ is a ring homomorphism.
 (b) The map $\text{gch} : \mathbf{GR} \rightarrow \mathbf{A}_X$ is injective.
 (c) \mathbf{GR} is a subring of $\mathbf{K}(\mathcal{C}_q)$.

Proof: Put $\gamma = \gamma^+ - \gamma^-$, where $\gamma^\pm = \sum_k q^{n_k^\pm} \omega_{i_k} \in X^+$. The eigenvalue of \mathbf{h}_{ir} on V_γ is $r^{-1}[r] \sum_{i_k=i} (q^{r n_k^+} - q^{r n_k^-})$, see [FM, (2.11)] for instance. It is known that $\Delta(\mathbf{h}_{ir}) = \mathbf{h}_{ir} \otimes 1 + 1 \otimes \mathbf{h}_{ir}$ modulo the linear span of elements $m_0 m^- m^0 \otimes n^0 n^+$, where m_0 (resp. m^-, m^0, n^0, n^+) is a monomial in the generators \mathbf{k}_i^\pm (resp. $\mathbf{x}_{is}^-, \mathbf{h}_{is}, \mathbf{h}_{is}, \mathbf{x}_{is}^+$) such that m^-, n^+ have a non-zero degree, see [D]. Thus, the weight γ subspace in $V' \otimes V''$ is

$$\bigoplus_{\gamma=\gamma'+\gamma''} (V_{\gamma'}' \otimes V_{\gamma''}'').$$

Then, Claim (a) follows from (2.3.1) and (2.4.1). Claim (b) is obvious. Claim (c) is proved in Theorem 4.3. \square

For a future use we introduce the following sets

$$\Lambda(\gamma) = \{\eta \in Y^+ \mid W(\gamma)_{\gamma-\eta} \neq \{0\}\}, \quad \Lambda^+(\gamma) = \{\eta \in \Lambda(\gamma) \mid \gamma \succeq \eta\}.$$

Remark. The map gch appeared first in [N3]. By [Vr], the same construction holds for Yangians. The specialization of gch at $v = 1$ first appeared in [Kn], for Yangians. The case of quantum affine algebras was done in [FR].

Example. We give a few computations in the case $\mathfrak{g} = \mathfrak{sl}_2$. To simplify we omit ω_1 : we write q^n instead of $q^n \omega_1$. We get

$$e^{q^m} \cdot e^{q^n} = v^t e^{q^m + q^n} \in \mathbf{A}_X,$$

where $t = 0$ if $n - m$ is zero or odd, and $t = (-1)^\ell$ if $n - m = 2\ell$ with $\ell < 0$. We have

$$\text{gch}V(q^n) = e^{q^n} + e^{-q^{n+2}} \in \mathbf{A}_X,$$

and

$$W(q^n + q^{n-2}) = v^{-\langle q^n, q^{n-2} \rangle} V(q^n) \otimes V(q^{n-2}), \quad W(kq^n) = V(q^n)^{\otimes k} \in \mathbf{GR}.$$

Thus,

$$\text{gch}W(q^n + q^{n-2}) = e^{q^n + q^{n-2}} + e^{q^{n-2} - q^{n+2}} + e^{-q^n - q^{n+2}} + v,$$

$$\text{gch}W(kq^n) = \sum_{i=0}^k \begin{bmatrix} k \\ i \end{bmatrix}_v e^{iq^n - (k-i)q^{n+2}},$$

where $\begin{bmatrix} k \\ i \end{bmatrix}_v$ is the v -binomial coefficient. Note that our normalizations are different from [N3] (we use $v = t^{-1}$).

3. REMINDER ON QUIVER VARIETIES

3.1. Consider the graph such that : I is the set of vertices, and there are $2\delta_{ij} - a_{ij}$ edges between $i, j \in I$. Each edge is endowed with the two possible orientations. The corresponding set of arrows is denoted by H . If $h \in H$ let h' and h'' the incoming and the outgoing vertex of h . Let $\bar{h} \in H$ denote the arrow opposite to h . Fix two I -graded finite dimensional complex vector spaces V, W of graded dimension $(a_i), (\ell_i)$. Let us fix once for all the following convention : the dimension of the graded vector space V is identified with the root $\alpha = \sum_{i \in I} a_i \alpha_i \in Q^+$ while the dimension of W is identified with the weight $\lambda = \sum_i \ell_i \omega_i \in P^+$. Set

$$E(V, W) = \bigoplus_{h \in H} M_{\ell_{h'}, a_{h''}}(\mathbb{C}), \quad L(V, W) = \bigoplus_{i \in I} M_{\ell_i, a_i}(\mathbb{C}),$$

$$M_{\lambda\alpha} = E(V, V) \oplus L(W, V) \oplus L(V, W).$$

For any $(B, p, q) \in M_{\lambda\alpha}$ let B_h be the component of B in $\text{Hom}(V_{h''}, V_{h'})$ and set

$$m_{\lambda\alpha}(B, p, q) = \sum_h \varepsilon(h) B_h B_{\bar{h}} + pq \in L(V, V),$$

where ε is a function $\varepsilon : H \rightarrow \mathbb{C}^\times$ such that $\varepsilon(h) + \varepsilon(\bar{h}) = 0$. A triple $(B, p, q) \in m_{\lambda\alpha}^{-1}(0)$ is \spadesuit -stable if there is no nontrivial B -invariant subspace of $\text{Ker } q$. Let $m_{\lambda\alpha}^{-1}(0)^\spadesuit$ be the subset of \spadesuit -stable triples. The group $\mathbb{C}^\times \times G_\lambda \times G_\alpha$ acts on $M_{\lambda\alpha}$ by

$$(z, g_\lambda, g_\alpha) \cdot (B, p, q) = (zg_\alpha B g_\alpha^{-1}, zg_\alpha p g_\alpha^{-1}, zg_\lambda q g_\alpha^{-1}).$$

The action of G_α on the subset $m_{\lambda\alpha}^{-1}(0)^\spadesuit$ is free. Consider the varieties

$$Q_{\lambda\alpha} = \text{Proj} \left(\bigoplus_{n \geq 0} A_n \right) \quad \text{and} \quad N_{\lambda\alpha} = m_{\lambda\alpha}^{-1}(0) // G_\alpha,$$

where $//$ is the categorical quotient, and

$$A_n = \{ f \in \mathbb{C}[m_{\lambda\alpha}^{-1}(0)] \mid f(g_\alpha \cdot (B, p, q)) = (\det g_\alpha)^{-n} f(B, p, q) \}.$$

The variety $Q_{\lambda\alpha}$ is smooth and there is a bijection $Q_{\lambda\alpha} \simeq m_{\lambda\alpha}^{-1}(0)^\spadesuit / G_\alpha$.

3.2. Let $\pi_{\lambda\alpha} : Q_{\lambda\alpha} \rightarrow N_{\lambda\alpha}$ be the affinization map. It is a proper map. Put $d_{\lambda\alpha} = \dim Q_{\lambda\alpha}$. It is known that $d_{\lambda\alpha} = (\alpha | 2\lambda - \alpha)$. If $\alpha \geq \beta$ the extension by zero of representations of the quiver gives a closed embedding $N_{\lambda\beta} \hookrightarrow N_{\lambda\alpha}$. Set $N_\lambda = \bigcup_\alpha N_{\lambda\alpha}$, $Q_\lambda = \bigsqcup_\alpha Q_{\lambda\alpha}$, $F_\lambda = \bigsqcup_\alpha F_{\lambda\alpha}$, where $F_{\lambda\alpha} = \pi_{\lambda\alpha}^{-1}(0)$. A triple $(B, p, q) \in m_{\lambda\alpha}^{-1}(0)$ is regular if it is \spadesuit -stable and its G_α -orbit is closed. Let $m_{\lambda\alpha}^{-1}(0)^\heartsuit \subseteq m_{\lambda\alpha}^{-1}(0)^\spadesuit$ be the subset of regular triples. Let $Q_{\lambda\alpha}^\heartsuit = m_{\lambda\alpha}^{-1}(0)^\heartsuit / G_\alpha$ and $N_{\lambda\alpha}^\heartsuit = m_{\lambda\alpha}^{-1}(0)^\heartsuit // G_\alpha$ be the corresponding open subsets in $Q_{\lambda\alpha}, N_{\lambda\alpha}$. The map $\pi_{\lambda\alpha}$ gives an isomorphism $Q_{\lambda\alpha}^\heartsuit \xrightarrow{\sim} N_{\lambda\alpha}^\heartsuit$. It is proved in [N1], [N2] that

- $N_{\lambda\alpha}^\heartsuit \neq \emptyset \iff \alpha \in \bigwedge^+(\lambda)$, and $Q_{\lambda\alpha} \neq \emptyset \iff \alpha \in \bigwedge(\lambda)$,
- $N_\lambda = \bigsqcup_\alpha N_{\lambda\alpha}^\heartsuit$, and $N_{\lambda\beta}^\heartsuit \subseteq \overline{N_{\lambda\alpha}^\heartsuit} \iff \alpha \geq \beta$.

3.3. The fixpoint set of a bijection $\phi : X \xrightarrow{\sim} X$ is denoted by X^ϕ . The group $\mathbb{C}^\times \times G_\lambda$ acts on $Q_{\lambda\alpha}$, $N_{\lambda\alpha}$. For any $k \in \mathbb{Z}$ and $(\gamma, \eta) \in G_\lambda^\vee \times G_\alpha^\vee$ we set

$$Q_{\gamma\eta,k} = (G_\alpha \cdot m_{\lambda\alpha}^{-1}(0)^{\blacklozenge, (q^k, \gamma, \eta)}) / G_\alpha, \quad Q_{\gamma,k} = Q_\lambda^{(q^k, \gamma)}, \quad N_{\gamma,k} = N_\lambda^{(q^k, \gamma)}.$$

It is known that $Q_{\gamma\eta,k}$ is either empty or a connected component of $Q_{\gamma,k}$. Let $\pi_{\gamma,k} : Q_{\gamma,k} \rightarrow N_{\gamma,k}$ be the restriction of the map π_λ . We set $F_{\gamma,k} = \pi_{\gamma,k}^{-1}(0)$, $F_{\gamma\eta,k} = F_{\gamma,k} \cap Q_{\gamma\eta,k}$, $Q_{\gamma\eta,k}^\heartsuit = Q_{\gamma\eta,k} \cap Q_{\lambda\alpha}^\heartsuit$, $N_{\gamma\eta,k}^\heartsuit = \pi_\gamma(Q_{\gamma\eta,k}^\heartsuit)$. The restriction of $\pi_{\gamma,k}$ to $Q_{\gamma\eta,k}^\heartsuit$ is an isomorphism onto $N_{\gamma\eta,k}^\heartsuit$. It is proved in [N2] that

- $Q_{\gamma,k} = \bigsqcup_\eta Q_{\gamma\eta,k}$, $N_{\gamma,k} = \bigsqcup_\eta N_{\gamma\eta,k}^\heartsuit$, and $Q_{\gamma\eta,k}$ is connected (or empty),
- the set $N_{\gamma\eta,k}^\heartsuit$ depends only on the conjugacy classes of γ, η ,
- $N_{\gamma\eta,1}^\heartsuit \neq \emptyset \iff \eta \in \Lambda^+(\gamma)$ and $Q_{\gamma\eta,1} \neq \emptyset \iff \eta \in \Lambda(\gamma)$.

To simplify, hereafter we set $Q_{\gamma\eta} = Q_{\gamma\eta,1}$, $Q_\gamma = Q_{\gamma,1}$, $N_\gamma = N_{\gamma,1}$, etc. Put

$$d_{\gamma\eta} = (\bar{\eta} \mid [2]\gamma - q\eta)_0.$$

If $Q_{\gamma\eta} \neq \emptyset$ then $d_{\gamma\eta} = \dim Q_{\gamma\eta}$, see [N2, (4.1.6)].

3.4. If an algebraic group G acts on a variety X , and if $\phi \in G^\vee$, we put

$$X^{+\phi} = \{x \in X \mid \lim_{z \rightarrow 0} \phi(z) \cdot x \in X^\phi\}, \quad X^{-\phi} = \{x \in X \mid \lim_{z \rightarrow \infty} \phi(z) \cdot x \in X^\phi\}.$$

For any $k \in \mathbb{Z}$, $\gamma \in G_\lambda^\vee$, $\tau \in (\mathbb{C}^\times \times G_\lambda)^\vee$ we have the commutative diagram

$$\begin{array}{ccccc} Q_{\gamma,k} & \xleftarrow{\tilde{\iota}_\pm} & Q_{\gamma,k}^{\pm\tau} & \xrightarrow{\tilde{\kappa}_\pm} & Q_{\gamma,k}^\tau \\ \downarrow & & \downarrow & & \downarrow \\ N_{\gamma,k} & \xleftarrow{\iota_\pm} & N_{\gamma,k}^{\pm\tau} & \xrightarrow{\kappa_\pm} & N_{\gamma,k}^\tau \end{array}$$

where $\tilde{\iota}_\pm, \iota_\pm$ are the embeddings, and $\tilde{\kappa}_\pm, \kappa_\pm$ are the obvious projections. Since the map $\pi_{\gamma,k}$ is proper the left square is Cartesian.

Remark. The maps $\tilde{\iota}_\pm, \iota_\pm$ are closed embeddings. We have $Q_{\gamma,k}^{\pm\tau} = \pi_{\gamma,k}^{-1}(N_{\gamma,k}^{\pm\tau})$ since $\pi_{\gamma,k}$ is a proper map. Thus, it is sufficient to consider the case of ι_\pm . From [L2], we can fix a finite set of generators of the ring $\mathbb{C}[m_{\lambda\alpha}^{-1}(0)]^{G_\alpha}$ consisting of eigenvectors of the group $\tau(\mathbb{C}^\times) \times \gamma(\mathbb{C}^\times) \subset \mathbb{C}^\times \times G_\lambda$. These generators give a $\tau(\mathbb{C}^\times)$ -equivariant closed embedding of the variety $N_{\gamma,k}$ in a finite dimensional representation of $\tau(\mathbb{C}^\times)$. But $X^{\pm\phi}$ is a closed subset of X in the particular case where X is a representation of the one-parameter subgroup ϕ . Thus $N_{\gamma,k}^{\pm\tau}$ is a closed subset of $N_{\gamma,k}$.

3.5. Fix $\lambda', \lambda'' \in P^+$, fix I -graded vector spaces W', W'' of dimension λ', λ'' , and fix $\gamma' \in G_{\lambda'}^\vee$, $\gamma'' \in G_{\lambda''}^\vee$. Put $\gamma = \gamma' + \gamma''$, $\lambda = \lambda' + \lambda''$, $W = W' \oplus W''$ and $\tau = q \cdot \text{Id}_{W'} \oplus \text{Id}_{W''}$.

Lemma 1. (a) The direct sum of representations of the quiver gives an isomorphism $Q_{\gamma',k} \times Q_{\gamma'',k} \simeq Q_{\gamma,k}^\tau$, and a map $\phi : N_{\gamma',k} \times N_{\gamma'',k} \rightarrow N_{\gamma,k}^\tau$.

(b) The map ϕ is finite, bijective and is compatible with the stratifications.

Proof: The first claim is well-known, see [VV, Lemma 4.4] for instance. Let $\phi : m_{\lambda'\alpha'}^{-1}(0) \times m_{\lambda''\alpha''}^{-1}(0) \rightarrow m_{\lambda\alpha}^{-1}(0)$ be the direct sum of representations of the quiver in

§3.1. The induced map $N_{\lambda'\alpha'} \times N_{\lambda''\alpha''} \rightarrow N_{\lambda\alpha}$ is a morphism of algebraic varieties. We have

$$\phi(m_{\lambda'\alpha'}^{-1}(0)^\heartsuit \times m_{\lambda''\alpha''}^{-1}(0)^\heartsuit) \subset m_{\lambda\alpha}^{-1}(0)^\heartsuit,$$

since a triple $(B, p, q) \in m_{\lambda\alpha}^{-1}(0)$ is regular if and only if it is stable and costable (i.e. there is no proper B -invariant subspace of V containing $\text{Im } p$), see [L2]. Fix $\eta' \in G_{\alpha'}^\vee$, $\eta'' \in G_{\alpha''}^\vee$ such that $\eta = \eta' + \eta''$. By the first part, ϕ gives an isomorphism $N_{\gamma'\eta',k}^\heartsuit \times N_{\gamma''\eta'',k}^\heartsuit \xrightarrow{\sim} (N_{\gamma\eta,k}^\heartsuit)^\tau$. In particular it induces a bijection

$$N_{\gamma',k} \times N_{\gamma'',k} = \bigsqcup_{\eta',\eta''} N_{\gamma'\eta',k}^\heartsuit \times N_{\gamma''\eta'',k}^\heartsuit \xrightarrow{\sim} \bigsqcup_{\eta} (N_{\gamma\eta,k}^\heartsuit)^\tau = N_{\gamma,k}^\tau,$$

which is compatible with the stratifications. This map is clearly affine, since $N_{\lambda\alpha}$ is an affine variety. Thus it is finite. \square

If $k = 1$ we get

$$\begin{array}{ccccccc} Q_\gamma & \xleftarrow{\tilde{\iota}^\pm} & Q_\gamma^{\pm\tau} & \xrightarrow{\tilde{\kappa}^\pm} & Q_\gamma^\tau & \simeq & Q_{\gamma'} \times Q_{\gamma''} \\ \pi_\gamma \downarrow & \square & \downarrow & & \downarrow & & \downarrow \\ N_\gamma & \xleftarrow{\iota^\pm} & N_\gamma^{\pm\tau} & \xrightarrow{\kappa^\pm} & N_\gamma^\tau & \xleftarrow{\phi} & N_{\gamma'} \times N_{\gamma''}. \end{array}$$

Fix $\eta' \in G_{\alpha'}^\vee$, $\eta'' \in G_{\alpha''}^\vee$. Let $\kappa_{\eta'\eta''}^\pm$ be the relative dimension of the map $\tilde{\kappa}_\pm$ above the component $Q_{\gamma'\eta'} \times Q_{\gamma''\eta''}$. Set $\eta = \eta' + \eta''$.

Lemma 2. *We have*

- (a) $\kappa_{\eta'\eta''}^+ + \kappa_{\eta'\eta''}^- = d_{\gamma\eta} - d_{\gamma'\eta'} - d_{\gamma''\eta''}$,
- (b) $\kappa_{\eta'\eta''}^\pm = \kappa_{\eta''\eta'}^\mp$,
- (c) *If $\delta' \in \Lambda^+(\gamma')$, $\delta'' \in \Lambda^+(\gamma'')$ are such that $\eta' \succeq \delta'$, $\eta'' \succeq \delta''$, then*

$$\varepsilon_{\gamma'\gamma''} - \varepsilon_{\gamma'-\delta',\gamma''-\delta''} = \kappa_{\eta'\eta''}^\pm - \kappa_{\eta'-\delta',\eta''-\delta''}^\pm = \kappa_{\delta'\delta''}^\pm.$$

Proof: Part (a) is immediate. Let us check Part (b). The one-parameter subgroup $q \cdot \text{Id}_{W'} \oplus \text{Id}_{W''}$ acts fiberwise on the normal bundle to $Q_{\gamma'\eta'} \times Q_{\gamma''\eta''}$ in $Q_{\gamma\eta}$. By definition $\kappa_{\eta'\eta''}^\pm$ is the dimension of the attracting (resp. repulsing) subbundle. The class in equivariant K -theory of the tangent bundle to $Q_{\gamma\eta}$ is given in [N1, §4.1]. We get

$$(1) \quad \kappa_{\eta'\eta''}^+ = (\bar{\eta}' | q^{-1}\gamma'')_0 + (\bar{\eta}'' | q\gamma')_0 - (\bar{\eta}'' | q\Omega(\eta'))_0,$$

and $\kappa_{\eta'\eta''}^- = \kappa_{\eta''\eta'}^+$. Observe that

$$(\Omega^{-1}(\gamma) | \Omega(\eta)) = (\eta | \gamma), \quad \forall \gamma \in X, \eta \in Y.$$

Part (c) is proved by a direct computation using (3.5.1) and

$$\begin{aligned} \varepsilon_{\gamma'\gamma''} - \varepsilon_{\gamma'-\delta',\gamma''-\delta''} &= (q^{-1}\Omega^{-1}(\bar{\gamma}') | \gamma'')_0 - (q^{-1}\Omega^{-1}(\bar{\gamma}') - q^{-1}\bar{\delta}' | \gamma'' - \Omega(\delta''))_0 \\ &= (q^{-1}\Omega^{-1}(\bar{\gamma}') | \Omega(\delta''))_0 + (q^{-1}\bar{\delta}' | \gamma'')_0 - (q^{-1}\bar{\delta}' | \Omega(\delta''))_0 \\ &= (q\bar{\delta}'' | \gamma')_0 + (q^{-1}\bar{\delta}' | \gamma'')_0 - (q\bar{\delta}'' | \Omega(\delta'))_0. \end{aligned}$$

\square

4. THE PRODUCT

4.1. For any complex algebraic variety X , let $\mathcal{D}(X)$ be the bounded derived category of complexes of constructible sheaves of \mathbb{C} -vector spaces on X . For any irreducible local system ϕ on a locally closed set $Y \subset X$, let $IC(Y, \phi)$ be the corresponding intersection cohomology complex. Let \mathbb{C}_Y be the constant sheaf on Y . We set $IC(Y) = IC(Y, \mathbb{C}_Y)$. Recall that the direct image of a simple perverse sheaf by a finite bijective map is still a simple perverse sheaf. Let \mathbb{D} denote the Verdier duality.

Fix $\gamma, \gamma' \in X^+$, $\eta, \eta' \in Y^+$. Fix λ, α such that $\gamma \in G_\lambda^{\vee, \text{ad}}$, $\eta \in G_\alpha^{\vee, \text{ad}}$. Hereafter we may identify a cocharacter in G_λ^\vee , G_α^\vee , and its conjugacy class in X^+ , Y^+ . Let $\mathcal{D}(N_\gamma)^\heartsuit$, $\mathcal{D}(N_\gamma \times N_{\gamma'})^\heartsuit$ be the full subcategories of $\mathcal{D}(N_\gamma)$, $\mathcal{D}(N_\gamma \times N_{\gamma'})$ consisting of all complexes which are constructible with respect to the stratification in §3.3. Set $IC_{\gamma\eta} = IC(N_{\gamma\eta}^\heartsuit)$, $\mathbb{C}_{\gamma\eta} = \mathbb{C}_{N_{\gamma\eta}^\heartsuit}[d_{\gamma\eta}]$ for any $\eta \in \Lambda^+(\gamma)$, and $L_{\gamma\eta} = \pi_{\gamma!} \mathbb{C}_{Q_{\gamma\eta}}[d_{\gamma\eta}]$ for any $\eta \in \Lambda(\gamma)$. Let \mathcal{Q}_γ , $\mathcal{Q}_{\gamma\gamma'}$ be the full subcategories of $\mathcal{D}(N_\gamma)^\heartsuit$, $\mathcal{D}(N_\gamma \times N_{\gamma'})^\heartsuit$ consisting of all complexes which are isomorphic to finite direct sums of the sheaves $IC_{\gamma\eta}[k]$, $IC_{\gamma\eta}[k] \boxtimes IC_{\gamma'\eta'}[k']$, $k, k' \in \mathbb{Z}$. The complex $L_{\gamma\eta}$ belongs to $\text{Ob}(\mathcal{Q}_\gamma)$, see [N2, Theorem 14.3.2]. If $\gamma', \gamma'', \iota_\pm, \kappa_\pm, \tau$ are as in §3.5, we have the functor

$$\text{res}_{\gamma'\gamma''}^\pm = \kappa_{\pm!} \iota_\pm^* : \mathcal{D}(N_\gamma)^\heartsuit \rightarrow \mathcal{D}(N_\gamma^\tau)^\heartsuit.$$

Lemma. *We have*

- (a) $\text{res}_{\gamma'\gamma''}^\pm(L_{\gamma\eta}) = \bigoplus_{\eta=\eta'+\eta''} \phi_!(L_{\gamma'\eta'} \boxtimes L_{\gamma''\eta''})[\kappa_{\eta'\eta''}^\mp - \kappa_{\eta'\eta''}^\pm]$,
- (b) $\mathbb{D} \circ \text{res}_{\gamma'\gamma''}^\pm = \text{res}_{\gamma'\gamma''}^\mp \circ \mathbb{D}$, and $\text{res}_{\gamma'\gamma''}^\pm = \text{res}_{\gamma''\gamma'}^\mp$.
- (c) *For any complex $P \in \text{Ob}(\mathcal{Q}_\gamma)$ there is a complex $P' \in \text{Ob}(\mathcal{Q}_{\gamma'\gamma''})$ such that $\text{res}_{\gamma'\gamma''}^\pm(P) \simeq \phi_!(P')$. The complex P' is unique up to isomorphism.*

Proof: By base change, the diagram in §3.5 gives

$$\text{res}_{\gamma'\gamma''}^\pm(L_{\gamma\eta}) = \pi_{\gamma!} \tilde{\kappa}_{\pm!} \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\gamma\eta}}[d_{\gamma\eta}].$$

From [L1, 8.1.6] the complex $\pi_{\gamma!} \tilde{\kappa}_{\pm!} \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\gamma\eta}}$ is semi-simple, and there are short exact sequences of perverse sheaves

$$0 \rightarrow {}^p H^n(f_j)_! \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\gamma\eta}} \rightarrow {}^p H^n(f_{\leq j})_! \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\gamma\eta}} \rightarrow {}^p H^n(f_{\leq j-1})_! \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\gamma\eta}} \rightarrow 0,$$

where ${}^p H^n$ is the perverse cohomology, and f_j (resp. $f_{\leq j}$) is the restriction of the map $\pi_\gamma \tilde{\kappa}_\pm$ to the union of all subvarieties

$$\tilde{\kappa}_\pm^{-1}(Q_{\gamma'\eta'} \times Q_{\gamma''\eta''}) \subset Q_\gamma^{\pm\tau}$$

of dimension j (resp. $\leq j$). We have also

$$\pi_{\gamma!} \tilde{\kappa}_{\pm!} \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\gamma\eta}}[d_{\gamma\eta}] = \phi_!(L_{\gamma'\eta'} \boxtimes L_{\gamma''\eta''})[d_{\gamma\eta} - 2\kappa_{\eta'\eta''}^\pm].$$

Thus, Claim (a) follows from Lemma 3.5.2.(a). Claim (b) is due to the auto-duality of $L_{\gamma\eta}$, since the map π_γ is proper, and Lemma 3.5.2.(b). The first part of Claim (c) follows from Claim (a), since a direct summand of a complex in \mathcal{Q}_γ belongs to \mathcal{Q}_γ . The second part of Claim (c) is due to Lemma 3.5.1.(b). \square

4.2. Let \mathcal{K}_γ be the \mathbb{A} -module with one generator for each isomorphism class of object of \mathcal{Q}_γ , with relations $P + P' = P''$ if the complex P'' is isomorphic to $P \oplus P'$, and $P = vP'$ if the complex P is isomorphic to $P'[1]$. The elements $IC_{\gamma\eta}$, with $\eta \in \Lambda^+(\gamma)$, form a \mathbb{A} -basis of \mathcal{K}_γ . Let $\text{res}_{\gamma'\gamma''}$ be the \mathbb{A} -linear map $\mathcal{K}_\gamma \rightarrow \mathcal{K}_{\gamma'} \otimes \mathcal{K}_{\gamma''}$ such that

$$\text{res}_{\gamma'\gamma''}(P) = v^{\langle \gamma', \gamma'' \rangle} \sum_i P'_i \otimes P''_i$$

where $\text{res}_{\gamma'\gamma''}^+(P) = \bigoplus_i \phi_!(P'_i \boxtimes P''_i)$. It is well-defined and unique by Lemma 4.1.(c).

Lemma 1. (a) In \mathcal{K}_γ we have

$$L_{\gamma\eta} = \sum_\delta \text{gdm} V(\gamma - \delta)_{\gamma-\eta} IC_{\gamma\delta}.$$

In particular, the elements $L_{\gamma\eta}$, with $\eta \in \Lambda^+(\gamma)$, form a \mathbb{A} -basis of \mathcal{K}_γ .

(b) If $\delta \in \Lambda^+(\gamma)$ there is a unique surjective map $\mathcal{K}_\gamma \rightarrow \mathcal{K}_{\gamma-\delta}$ such that $L_{\gamma\eta} \mapsto L_{\gamma-\delta, \eta-\delta}$ if $\eta \in \Lambda(\gamma)$, $\eta \succeq \delta$, and $L_{\gamma\eta} \mapsto 0$ else.

(c) If $\delta \in \Lambda^+(\gamma)$, $\delta' \in \Lambda^+(\gamma')$, $\delta'' \in \Lambda^+(\gamma'')$, $\delta = \delta' + \delta''$, the square

$$\begin{array}{ccc} \mathcal{K}_\gamma & \xrightarrow{\text{res}} & \mathcal{K}_{\gamma'} \otimes \mathcal{K}_{\gamma''} \\ \downarrow & & \downarrow \\ \mathcal{K}_{\gamma-\delta} & \xrightarrow{\text{res}} & \mathcal{K}_{\gamma'-\delta'} \otimes \mathcal{K}_{\gamma''-\delta''} \end{array}$$

is commutative.

Proof: Fix $\delta \in \Lambda^+(\gamma) \cap G_\beta^{\vee, \text{ad}}$ such that $\eta \succeq \delta$, and fix $x_\delta \in N_{\gamma\delta}^\heartsuit$. We have an isomorphism

$$W(\gamma - \delta)_{\gamma-\eta} \simeq \bigoplus_k H_k(F_{\gamma-\delta, \eta-\delta}) \simeq \bigoplus_k H_k(Q_{\gamma\eta} \cap \pi_\gamma^{-1}(x_\delta))$$

such that $w_{\gamma-\delta} \in H_0(F_{\gamma-\delta, 0})$, see [VV, Theorem 7.12], [N2, Theorems 3.3.2 and 7.4.1]. We first check that

$$(1) \quad \text{gdm} W(\gamma - \delta)_{\gamma-\eta} = \sum_k v^{d_{\gamma-\delta, \eta-\delta} - k} \dim H_k(Q_{\gamma\eta} \cap \pi_\gamma^{-1}(x_\delta)).$$

To simplify the notations, we may assume that $\delta = 0$, without loss of generalities. Let $C_{\lambda, \alpha + \alpha_i, \alpha} \subseteq Q_{\lambda, \alpha + \alpha_i} \times Q_{\lambda\alpha}$ be the set of pairs (x', x) such that x is a subrepresentation of x' . For any η, η' put

$$C_{\eta'\eta} = C_{\lambda, \alpha + \alpha_i, \alpha} \cap (Q_{\gamma\eta'} \times Q_{\gamma\eta}).$$

If $C_{\eta'\eta} \neq \emptyset$ then $\eta' = \eta + q^t \alpha_i$ for some $t \in \mathbb{Z}$. Set

$$d_{\eta'\eta} = \dim C_{\eta'\eta}, \quad e_{\eta'\eta} = d_{\gamma\eta} + d_{\gamma\eta'} - 2d_{\eta'\eta}.$$

Let \star be the convolution product in Borel-Moore homology, see [CG]. By definition, we have

$$H_{d_{\gamma\eta'} + d_{\gamma\eta} - e}^{BM}(C_{\eta'\eta}) \star H_{d_{\gamma\eta} - \ell}(F_{\gamma\eta}) \subseteq H_{d_{\gamma\eta'} - k}(F_{\gamma\eta'}), \quad k = \ell + e,$$

see [CG, Lemma 8.9.5]. Recall that $x_{ir}^{(t)}$ acts on $H_*(F_\gamma)$ by the \star -product by an element of the form

$$\sum_{\eta} (\theta_{\eta'\eta} \cup q^{rt} e^{r\omega_{\eta'\eta}}) \cap [C_{\eta'\eta}] \in H_*^{BM}(C_{\eta'\eta}),$$

where $\eta' = \eta + q^t \alpha_i$, $[C_{\eta'\eta}]$ is the fundamental class, $\omega_{\eta'\eta}, \theta_{\eta'\eta} \in H^{2*}(Q_{\gamma\eta'} \times Q_{\gamma\eta})$, $\deg \omega_{\eta'\eta} = 2$, and $\theta_{\eta'\eta}$ is invertible. Moreover, $\omega_{\eta'\eta}, \theta_{\eta'\eta}$ do not depend on r . More precisely, from [N2, (9.3.2), §13.4], we have

$$(2) \quad \theta_{\eta'\eta} = e^{k\omega_{\eta'\eta}} \cup (1 \otimes \nu_{\eta'\eta})$$

where $k \in \mathbb{Z}$ and $\nu_{\eta'\eta} \in H^{2*}(Q_{\gamma\eta})$ is invertible. Fix a non-zero $v \in H_0(F_{\gamma_0})$. The space $H_*(F_\gamma)$ is spanned by the elements $\mathbf{x}_{i_1 r_1}^- \cdots \mathbf{x}_{i_s r_s}^-(v)$. Thus, for any $\eta' \in Y^+ \setminus 0$, we get

$$H_*(F_{\gamma\eta'}) = \sum_{i,t,r} x_{ir}^{(t)} \star H_*(F_{\gamma\eta}),$$

where $\eta = \eta' - q^t \alpha_i$. Set $\psi_{ir}^{(t)} = \sum_{\eta} \omega_{\eta'\eta}^r \cap [C_{\eta'\eta}]$. Using (4.2.2) we get

$$H_*(F_{\gamma\eta'}) = \sum_{i,t,r} \psi_{ir}^{(t)} \star H_*(F_{\gamma\eta}).$$

The \star -product by $\psi_{ir}^{(t)}$ on $H_*(F_{\gamma\eta})$ is a homogeneous operator of degree $e_{\eta'\eta} + 2r \in \mathbb{Z}$. Thus,

$$H_{d_{\gamma\eta}-k}(F_{\gamma\eta'}) = \sum_{i,t,r} \psi_{ir}^{(t)} \star H_{d_{\gamma\eta}-\ell}(F_{\gamma\eta}),$$

where $k = e_{\eta'\eta} + \ell + 2r$. Set

$$F_\ell H_*(F_{\gamma\eta}) = \bigoplus_{\ell' \geq \ell} H_{d_{\gamma\eta}-\ell'}(F_{\gamma\eta}).$$

A direct computation gives

$$\phi_{ir}^{(t)} = \sum_{\eta} \theta_{\eta'\eta} \cap (e^{\omega_{\eta'\eta}} - 1)^r \cap [C_{\eta'\eta}],$$

where $\eta' = \eta + q^t \alpha_i$. Thus

$$(3) \quad F_k H_*(F_{\gamma\eta'}) = \sum_{i,t,r} \phi_{ir}^{(t)} \star F_\ell H_*(F_{\gamma\eta}),$$

where $k = e_{\eta'\eta} + \ell + 2r$.

The γ -fixed part of the complex [N2, (5.1.1)] is the normal bundle of $C_{\eta'\eta}$ in $Q_{\gamma\eta'} \times Q_{\gamma\eta}$. Thus

$$d_{\gamma\eta} + d_{\gamma\eta'} - d_{\eta'\eta} = (q\bar{\eta} + q^{-1}\bar{\eta}' \mid \gamma)_0 - (q\bar{\eta} \mid \Omega(\eta'))_0.$$

From $\eta' = \eta + q^t \alpha_i$, we get

$$e_{\eta'\eta} = (\bar{\eta}' - \bar{\eta} \mid q^{-1}(\gamma - \eta) - q(\gamma - \eta'))_0 = 1 + (\alpha_i \mid q^{-t}(q^{-1} - q)(\gamma - \eta'))_0.$$

Using (4.2.3) and §2.4 we get

$$F_\ell W(\gamma)_{\gamma-\eta} = F_\ell H_*(F_{\gamma\eta}).$$

The identity (4.2.1) follows.

To prove Lemma 4.2.1.(a) set $L_{\gamma\eta} = \bigoplus_{k, \delta \preceq \eta} M_{\delta k} \otimes IC_{\gamma\delta}[k]$. If $\gamma, \eta \succeq \delta$, let

$$\phi_{\delta k} : H_{d_{\gamma-\delta, \eta-\delta}-k}(F_{\gamma-\delta, \eta-\delta}) \rightarrow H^{d_{\gamma-\delta, \eta-\delta}+k}(F_{\gamma-\delta, \eta-\delta}),$$

be the composition of the chain of maps

$$H_{*-k}(F_{\gamma-\delta, \eta-\delta}) \rightarrow H_{*-k}^{BM}(Q_{\gamma-\delta, \eta-\delta}) \rightarrow H^{*+k}(Q_{\gamma-\delta, \eta-\delta}) \rightarrow H^{*+k}(F_{\gamma-\delta, \eta-\delta}).$$

A detailed analysis of the gradings in [N2, §14], [CG, §8] shows that $M_{\delta k} = \text{Im } \phi_{\delta k}$. Since $V(\gamma - \delta)_{\gamma-\eta} \simeq \bigoplus_k M_{\delta k}$, we get $\text{gdm} V(\gamma - \delta)_{\gamma-\eta} = \sum_k v^k \dim M_{\delta k}$.

Let us prove part (b). By [N2, Theorem 3.3.2] we have for any $\delta \in \bigwedge^+(\gamma)$

$$N_{\gamma-\delta, \eta-\delta}^\heartsuit = \emptyset \iff N_{\gamma\eta}^\heartsuit = \emptyset, \quad Q_{\gamma-\delta, \eta-\delta} = \emptyset \iff Q_{\gamma\eta} = \emptyset.$$

Thus, using §3.3 we get

$$(4) \quad \begin{aligned} \eta - \delta \in \bigwedge^+(\gamma - \delta) &\iff \eta \in \bigwedge^+(\gamma), \quad \eta \succeq \delta, \\ \eta - \delta \in \bigwedge(\gamma - \delta) &\iff \eta \in \bigwedge(\gamma), \quad \eta \succeq \delta. \end{aligned}$$

By (4.2.4) there is a unique surjective map $\mathcal{K}_\gamma \rightarrow \mathcal{K}_{\gamma-\delta}$ such that

$$IC_{\gamma\eta} \mapsto IC_{\gamma\eta} \quad \text{if } \eta \succeq \delta, \quad \text{and} \quad IC_{\gamma\eta} \mapsto 0 \quad \text{else.}$$

Using (4.2.4) again and Claim (a) of the lemma, we see that this map satisfies the requirements in Claim (b).

Set

$$\begin{aligned} A &= \kappa_{\eta'\eta''}^- - \kappa_{\eta'\eta''}^+ + \langle \gamma', \gamma'' \rangle, \\ B &= \kappa_{\eta'-\delta', \eta''-\delta''}^- - \kappa_{\eta'-\delta', \eta''-\delta''}^+ + \langle \gamma' - \delta', \gamma'' - \delta'' \rangle. \end{aligned}$$

Using Lemma 3.5.2.(b), (c) we get $A = B$. Thus, Claim (c) follows from Claim (b) and Lemma 4.1.(a). \square

4.3. Let $(\mathbf{b}_{\gamma\eta}), (\mathbf{c}_{\gamma\eta})$ be the bases of $\mathbf{GA}_\gamma = \text{Hom}_{\mathbb{A}}(\mathcal{K}_\gamma, \mathbb{A})$ dual to $(IC_{\gamma\eta}), (L_{\gamma\eta})$. Let $\otimes : \mathbf{GA}_{\gamma'} \otimes \mathbf{GA}_{\gamma''} \rightarrow \mathbf{GA}_{\gamma'+\gamma''}$ and $\theta : \mathbf{GA}_\gamma \rightarrow \mathbf{GA}_\gamma$ be the maps dual to $\text{res}_{\gamma'\gamma''}$ and \mathbb{D} . We consider the inductive system of \mathbb{A} -modules (\mathbf{GA}_γ) such that $\mathbf{b}_{\gamma\eta} \mapsto \mathbf{b}_{\gamma+\delta, \eta+\delta}$. Let $\mathbf{GA} = \varinjlim_\gamma \mathbf{GA}_\gamma$ be the limit. Let $\mathbf{b}_\gamma, \mathbf{c}_\gamma \in \mathbf{GA}$ be the images of the elements $\mathbf{b}_{\gamma 0}, \mathbf{c}_{\gamma 0} \in \mathbf{GA}_\gamma$.

Theorem. *The \mathbb{A} -module \mathbf{GR} is a subring of $\mathbf{K}(\mathcal{C}_q)$. The linear map such that $\mathbf{b}_\gamma \mapsto V(\gamma)$ is an algebra isomorphism $\mathbf{GA} \xrightarrow{\sim} \mathbf{GR}$. The map θ is a skew-linear antihomomorphism of \mathbf{GA} fixing the bases $\mathbf{B} = (\mathbf{b}_\gamma), \mathbf{C} = (\mathbf{c}_\gamma)$. For any γ, γ' we have*

$$\mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} \in \bigoplus_{\gamma''} \mathbb{N}[v^{-1}, v] \cdot \mathbf{b}_{\gamma''}.$$

Proof: The maps $\mathbb{D}, \text{res}_{\gamma'\gamma''}$ are compatible with the projective system (\mathcal{K}_γ) . The limit, denoted $(\mathcal{K}, \text{res})$, is a co-algebra with a skew-linear involution \mathbb{D} . By Lemma

4.2.1.(a), (b), the projective system maps $IC_{\gamma\eta}$ to $IC_{\gamma-\delta, \eta-\delta}$, for any $\eta \in \bigwedge^+(\gamma)$ such that $\eta \succeq \delta$. In \mathcal{K} we consider the elements $IC_\gamma = (IC_{\gamma+\delta, \delta})$, with $\gamma \in X^+$, and $L_\gamma = (L_{\gamma+\delta, \delta})$, with $\gamma \in X$. We have

$$\text{res}(L_\gamma) = \sum_{\gamma=\gamma'+\gamma''} v^{\langle \gamma', \gamma'' \rangle} L_{\gamma'} \otimes L_{\gamma''}.$$

Let \mathbf{A}_X^\vee be the \mathbb{A} -coalgebra with the \mathbb{A} -basis (\mathbf{a}_γ) , $\gamma \in X$, and the coproduct $\mathbf{a}_\gamma \mapsto \sum_{\gamma=\gamma'+\gamma''} v^{\langle \gamma', \gamma'' \rangle} \mathbf{a}_{\gamma'} \otimes \mathbf{a}_{\gamma''}$. The \mathbb{A} -linear map $\mathbf{A}_X^\vee \rightarrow (\mathcal{K}, \text{res})$ such that $\mathbf{a}_\gamma \mapsto L_\gamma$ is a surjective co-algebra homomorphism. By Lemma 4.2.1.(a) we have

$$L_\gamma = \sum_{\gamma' \in X^+} \text{gdm} V(\gamma')_\gamma \cdot IC_{\gamma'}, \quad \forall \gamma \in X.$$

The elements $IC_{\gamma'}$, $\gamma' \in X^+$, form a \mathbb{A} -basis of \mathcal{K} . Thus, the linear map

$$\psi : \mathbf{GA} \rightarrow \mathbf{A}_X, \quad \mathbf{b}_{\gamma'} \mapsto \sum_{\gamma \in X} \text{gdm} V(\gamma')_\gamma \cdot e^\gamma, \quad \forall \gamma' \in X^+,$$

is an injective ring homomorphism. Consider the linear map $\phi : \mathbf{GR} \rightarrow \mathbf{GA}$ such that $V(\gamma) \mapsto \mathbf{b}_\gamma$ for all $\gamma \in X^+$. We get the commutative square of linear maps

$$\begin{array}{ccc} \mathbf{GR} & \xrightarrow{\phi} & \mathbf{GA} \\ \downarrow & & \downarrow \psi \\ \mathbf{K}(\mathcal{C}_q) & \xrightarrow{\text{gch}} & \mathbf{A}_X \end{array}$$

where ψ, gch are ring homomorphisms, see Proposition 2.4.(a), the vertical maps are injective, and ϕ is invertible. Thus, \mathbf{GR} is a subring of $\mathbf{K}(\mathcal{C}_q)$ and ϕ is a ring homomorphism. If $\gamma' + \gamma'' = \gamma$ in X^+ , then

$$(\mathbb{D} \otimes \mathbb{D}) \circ \text{res}_{\gamma' \gamma''} \circ \mathbb{D} = \text{res}_{\gamma'' \gamma'}.$$

Thus θ is an antihomomorphism. □

If $\mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} = v^{\langle \gamma, \gamma' \rangle} \mathbf{b}_{\gamma+\gamma'}$, then the \mathbf{U} -module $V(\gamma) \otimes V(\gamma')$ is simple and isomorphic to $V(\gamma + \gamma')$. Conversely, if $V(\gamma) \otimes V(\gamma')$ is a simple \mathbf{U} -module it is isomorphic to $V(\gamma + \gamma')$. Then, the positivity in Theorem 4.3 implies that $\mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} \in v^{\mathbb{Z}} \mathbf{b}_{\gamma+\gamma'}$. Then, by (2.3.1) we get $\mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} = v^{\langle \gamma, \gamma' \rangle} \mathbf{b}_{\gamma+\gamma'}$. The following conjecture generalizes to all simply laced types the conjecture in [BZ] (for type A).

Conjecture. *The following statements are equivalent :*

$$\mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} \in v^{\mathbb{Z}} \mathbf{B}, \quad \mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} \in v^{\mathbb{Z}} \mathbf{b}_{\gamma'} \otimes \mathbf{b}_\gamma, \quad \text{and} \quad \mathbf{b}_\gamma \otimes \mathbf{b}_{\gamma'} = v^{\langle \gamma, \gamma' \rangle} \mathbf{b}_{\gamma+\gamma'}.$$

5. THE CLASSICAL CASE

5.1. Fix $\lambda, \lambda' \in P^+$. Let $\mathcal{D}(N_\lambda)^\heartsuit$, $\mathcal{D}(N_\lambda \times N_{\lambda'})^\heartsuit$ be the full subcategories of $\mathcal{D}(N_\lambda)$, $\mathcal{D}(N_\lambda \times N_{\lambda'})$ consisting of all complexes which are constructible with respect to the stratification in §3.2. Set $IC_{\lambda\alpha} = IC(N_{\lambda\alpha}^\heartsuit)$, $\mathbb{C}_{\lambda\alpha} = \mathbb{C}_{N_{\lambda\alpha}^\heartsuit}[d_{\lambda\alpha}]$ if $\alpha \in \bigwedge^+(\lambda)$, and set $L_{\lambda\alpha} = \pi_{\lambda\alpha}! \mathbb{C}_{Q_{\lambda\alpha}}[d_{\lambda\alpha}]$ if $\alpha \in \bigwedge(\lambda)$. Let \mathcal{P}_λ , $\mathcal{P}_{\lambda\lambda'}$ be the full subcategories of

$\mathcal{D}(N_\lambda)^\heartsuit$, $\mathcal{D}(N_\lambda \times N_{\lambda'})^\heartsuit$ consisting of all complexes which are isomorphic to finite direct sums of complexes of the form $IC_{\lambda\alpha}$, $IC_{\lambda\alpha} \boxtimes IC_{\lambda'\alpha'}$.

Assume that $\lambda = \lambda' + \lambda''$. Setting $k = 0$, $\gamma = \text{Id}_W$ in §3.5 we get the commutative diagram

$$\begin{array}{ccccccc} Q_\lambda & \xleftarrow{\tilde{\iota}^\pm} & Q_\lambda^{\pm\tau} & \xrightarrow{\tilde{\kappa}^\pm} & Q_\lambda^\tau & \simeq & Q_{\lambda'} \times Q_{\lambda''} \\ \downarrow & \square & \downarrow & & \downarrow & & \downarrow \\ N_\lambda & \xleftarrow{\iota^\pm} & N_\lambda^{\pm\tau} & \xrightarrow{\kappa^\pm} & N_\lambda^\tau & \xleftarrow{\phi} & N_{\lambda'} \times N_{\lambda''}. \end{array}$$

The restriction of the map $\tilde{\kappa}_\pm$ to $\tilde{\kappa}_\pm^{-1}(Q_{\lambda'\alpha'} \times Q_{\lambda''\alpha''})$ is a vector bundle of rank

$$(1) \quad (d_{\lambda\alpha} - d_{\lambda'\alpha'} - d_{\lambda''\alpha''})/2,$$

where $\alpha = \alpha' + \alpha''$. Indeed, let $T_{\lambda\tau}$ be the normal bundle to Q_λ^τ in Q_λ , and let $T_{\lambda\tau}^\pm$ be the restriction to Q_λ^τ of the relative tangent bundle to the map $\tilde{\kappa}_\pm$. The cocharacter τ acts on $T_{\lambda\tau}$ with non zero weights, and $T_{\lambda\tau}^\pm$ is the subbundle consisting of the positive (resp. negative) weights subspaces. Recall that Q_λ has a G_λ -invariant holomorphic symplectic form, see [N1, (3.3)]. Thus, the subvariety Q_λ^τ is symplectic, and the rank of $T_{\lambda\tau}$ is twice the rank of $T_{\lambda\tau}^\pm$.

Consider the functor

$$\text{res}_{\lambda'\lambda''}^\pm = \kappa_{\pm!} \iota_\pm^* : \mathcal{D}(N_\lambda)^\heartsuit \rightarrow \mathcal{D}(N_{\lambda'}^\tau)^\heartsuit.$$

For any $\mu \in P^+$ we set

$$V(\lambda', \lambda'')_\mu = \text{Hom}_{\mathfrak{g}}(V(\mu), V(\lambda') \otimes V(\lambda'')).$$

Lemma 1. *For any $\alpha \in \bigwedge(\lambda)$ we have*

- (a) $\text{res}_{\lambda'\lambda''}^+(L_{\lambda\alpha}) = \text{res}_{\lambda'\lambda''}^-(L_{\lambda\alpha}) = \bigoplus_{\alpha=\alpha'+\alpha''} \phi_!(L_{\lambda'\alpha'} \boxtimes L_{\lambda''\alpha''})$,
- (b) $\text{res}_{\lambda'\lambda''}^\pm(IC_{\lambda\alpha}) \simeq \bigoplus_{\alpha', \alpha''} V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda-\alpha} \otimes \phi_!(IC_{\lambda'\alpha'} \boxtimes IC_{\lambda''\alpha''})$,
- (c) $\text{res}_{\lambda'\lambda''}^\pm$ commutes to the Verdier duality.
- (d) For any complex $P \in \text{Ob}(\mathcal{P}_\lambda)$ there is a complex $P' \in \text{Ob}(\mathcal{P}_{\lambda'\lambda''})$ such that $\text{res}_{\lambda'\lambda''}^\pm(P) \simeq \phi_!(P')$.

Proof: Claim (a) is proved as Lemma 4.1, using (5.1.1). Using [N2, Theorem 15.3.2] we get an isomorphism

$$(2) \quad L_{\lambda\alpha} \simeq \bigoplus_{\beta \in Q^+} H_{\text{top}}(F_{\lambda-\beta, \alpha-\beta}) \otimes IC_{\lambda\beta}.$$

Using Part (a) and (5.1.2) we get

$$\begin{aligned} & \bigoplus_{\alpha \geq \beta} V(\lambda - \beta)_{\lambda-\alpha} \otimes \text{res}_{\lambda'\lambda''}^\pm(IC_{\lambda\beta}) \simeq \\ & \simeq \bigoplus_{\alpha \geq \beta} \bigoplus_{\beta', \beta''} V(\lambda - \beta)_{\lambda-\alpha} \otimes V(\lambda' - \beta', \lambda'' - \beta'')_{\lambda-\beta} \otimes \phi_!(IC_{\lambda'\beta'} \boxtimes IC_{\lambda''\beta''}), \end{aligned}$$

where the sum is over all $\beta', \beta'' \in Q^+$. An induction on β gives

$$\text{res}_{\lambda'\lambda''}^\pm(IC_{\lambda\beta}) \simeq \bigoplus_{\beta', \beta''} V(\lambda' - \beta', \lambda'' - \beta'')_{\lambda-\beta} \otimes \phi_!(IC_{\lambda'\beta'} \boxtimes IC_{\lambda''\beta''}).$$

□

By Lemma 3.5.1.(b), the functor $\phi_!$ is an equivalence from $\mathcal{P}_{\lambda', \lambda''}$ to a full subcategory of $\mathcal{D}(N_\lambda^\tau)^\heartsuit$. Composing $\text{res}_{\lambda', \lambda''}^\pm$ with a quasi-inverse to $\phi_!$ we get a functor $\text{res}_{\lambda', \lambda''} : \mathcal{P}_\lambda \rightarrow \mathcal{P}_{\lambda', \lambda''}$. Let \mathcal{Vec} be the category of finite dimensional complex vector spaces, and let $\mathcal{P}_\lambda^\circ$ be the category dual to \mathcal{P}_λ . We consider the following functors

$$\odot : \mathcal{P}_{\lambda'}^\circ \times \mathcal{P}_{\lambda''}^\circ \rightarrow \mathcal{P}_\lambda^\circ, \quad (P', P'') \mapsto \bigoplus_\alpha \text{Hom}_{\mathcal{P}_{\lambda', \lambda''}}(\text{res}_{\lambda', \lambda''}(IC_{\lambda\alpha}), P' \boxtimes P'') \otimes IC_{\lambda\alpha},$$

$$\Phi_\lambda : \mathcal{P}_\lambda^\circ \rightarrow \mathcal{Vec}, \quad P \mapsto \text{Hom}_{\mathcal{P}_\lambda}(P, \bigoplus_\alpha L_{\lambda\alpha}),$$

$$p_{\lambda\beta} : \mathcal{P}_\lambda^\circ \rightarrow \mathcal{P}_{\lambda-\beta}^\circ, \quad P \mapsto \bigoplus_{\alpha \geq \beta} \text{Hom}_{\mathcal{P}_\lambda}(IC_{\lambda\alpha}, P) \otimes IC_{\lambda-\beta, \alpha-\beta},$$

where $\beta \in \bigwedge^+(\lambda)$. Note that [N2, Theorem 3.3.2] and §3.2 give

$$\alpha - \beta \in \bigwedge^+(\lambda - \beta) \iff \alpha \in \bigwedge^+(\lambda), \quad \alpha \geq \beta,$$

and similarly with $\bigwedge(\lambda)$. By (5.1.2) we have

$$p_{\lambda\beta}(L_{\lambda\alpha}) \simeq \begin{cases} L_{\lambda-\beta, \alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else.} \end{cases}$$

We define a new category \mathcal{P}° as follows. Objects of \mathcal{P}° are collections $P = (P_\lambda, \gamma_{\lambda\beta})$, where $\lambda \in P^+$, $\beta \in \bigwedge^+(\lambda) \setminus \{0\}$, $P_\lambda \in \text{Ob}(\mathcal{P}_\lambda)$ and

$$\gamma_{\lambda\beta} \in \text{Isom}_{\mathcal{P}_{\lambda-\beta}}(P_{\lambda-\beta}, p_{\lambda\beta}(P_\lambda))$$

are isomorphisms satisfying the obvious chain condition. Morphisms $P' \rightarrow P''$ are collections $(\phi_\lambda) \in \prod_\lambda \text{Hom}_{\mathcal{P}_\lambda}(P'_\lambda, P''_\lambda)$ such that

$$\gamma'_{\lambda\beta} \circ \phi_{\lambda-\beta} = p_{\lambda\beta}(\phi_\lambda) \circ \gamma''_{\lambda\beta} \in \text{Hom}_{\mathcal{P}_{\lambda-\beta}}(P''_{\lambda-\beta}, p_{\lambda\beta}(P'_\lambda)).$$

Lemma 2. Fix $\beta \in \bigwedge^+(\lambda)$, $\beta' \in \bigwedge^+(\lambda')$, $\beta'' \in \bigwedge^+(\lambda'')$ such that $\beta = \beta' + \beta''$. For any $P, P', P'' \in \text{Ob}(\mathcal{P}^\circ)$ we have natural embeddings

$$\Phi_{\lambda-\beta}(P_{\lambda-\beta}) \subset \Phi_\lambda(P_\lambda), \quad P'_{\lambda'-\beta'} \odot P''_{\lambda''-\beta''} \subset p_{\lambda\beta}(P'_{\lambda'} \odot P''_{\lambda''}).$$

Moreover we have $\sum_{\beta', \beta''} P'_{\lambda'-\beta'} \odot P''_{\lambda''-\beta''} = p_{\lambda\beta}(P'_{\lambda'} \odot P''_{\lambda''})$.

Proof: Fix an isomorphism as in (5.1.2) for each $\alpha \in \bigwedge(\lambda)$. For any such α we get a morphism of functors

$$\bigoplus_{\alpha'} \text{Hom}(-, IC_{\lambda\alpha'}) \otimes \text{Hom}(IC_{\lambda-\beta, \alpha'-\beta}, L_{\lambda-\beta, \alpha-\beta}) \rightarrow \text{Hom}(-, L_{\lambda\alpha}).$$

By definition of Φ_λ , $p_{\lambda\beta}$ this morphism gives a morphism of functors $\Phi_{\lambda-\beta} \circ p_{\lambda\beta} \rightarrow \Phi_\lambda$. The morphism $\Phi_{\lambda-\beta}(P_{\lambda-\beta}) \rightarrow \Phi_\lambda(P_\lambda)$ is the composition of the isomorphism $\Phi_{\lambda-\beta}(\gamma_{\lambda\beta})$ and the morphism $\Phi_{\lambda-\beta} \circ p_{\lambda\beta} \rightarrow \Phi_\lambda$ above. Using (5.1.2) we get

$$\Phi_\lambda(IC_{\lambda\alpha}) \simeq V(\lambda - \alpha), \quad \Phi_{\lambda-\beta} \circ p_{\lambda\beta}(IC_{\lambda\alpha}) \simeq \begin{cases} V(\lambda - \alpha) & \text{if } \alpha \geq \beta, \\ 0 & \text{else.} \end{cases}$$

This proves Claim one. For any $\alpha' \in \bigwedge^+(\lambda')$, $\alpha'' \in \bigwedge^+(\lambda'')$ Lemma 5.1.1 gives an isomorphism of complexes

$$(3) \quad IC_{\lambda'\alpha'} \odot IC_{\lambda''\alpha''} \simeq \bigoplus_{\alpha} V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \otimes IC_{\lambda\alpha},$$

where the sum is over all $\alpha \in \bigwedge^+(\lambda)$ such that $V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \neq \{0\}$. Fix such a family of isomorphisms. It gives a morphism of functors $p_{\lambda'\beta'}(-) \odot p_{\lambda''\beta''}(-) \rightarrow p_{\lambda\beta}(- \odot -)$. The morphism $P'_{\lambda' - \beta'} \odot P''_{\lambda'' - \beta''} \rightarrow p_{\lambda\beta}(P'_{\lambda'} \odot P''_{\lambda''})$ is the composition of the isomorphism $\gamma'_{\lambda' - \beta'} \odot \gamma''_{\lambda'' - \beta''}$ and the morphism of functors $p_{\lambda'\beta'}(-) \odot p_{\lambda''\beta''}(-) \rightarrow p_{\lambda\beta}(- \odot -)$ above. Then, Claim two and three are consequences of the following identities. If $V(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda - \alpha} \neq \{0\}$, then $\alpha \geq \alpha' + \alpha''$, and thus

$$\alpha \geq \beta \quad \Leftarrow \quad \alpha' \geq \beta', \alpha'' \geq \beta'',$$

$$\alpha \geq \beta \quad \Rightarrow \quad \exists \beta', \beta'' \quad \text{s.t.} \quad \alpha' \geq \beta', \alpha'' \geq \beta'', \beta = \beta' + \beta''.$$

We are done. \square

By Lemma 5.1.2 the category \mathcal{P}° is endowed with the functors $\Phi : \mathcal{P}^\circ \rightarrow \mathcal{Vec}$, $\odot : \mathcal{P}^\circ \times \mathcal{P}^\circ \rightarrow \mathcal{P}^\circ$ such that

$$\Phi(P) = \lim_{\rightarrow \lambda} \Phi_\lambda(P_\lambda), \quad (P' \odot P'')_\lambda = \sum_{\lambda = \lambda' + \lambda''} P'_{\lambda'} \odot P''_{\lambda''}.$$

Then, (5.1.3) gives the following.

Lemma 3. *(\mathcal{P}°, \odot) is a tensor category, and Φ is a tensor functor.*

Let \mathbf{A} be the Grothendieck group of \mathcal{P}° . The functor \odot gives a product $\mathbf{A} \otimes \mathbf{A} \rightarrow \mathbf{A}$. Let $\mathbf{b}_\lambda, \mathbf{c}_\lambda$ be the classes in \mathbf{A} of the objects of \mathcal{P}° associated to the families $(IC_{\lambda+\beta, \beta}), (L_{\lambda+\beta, \beta})$. Then $(\mathbf{b}_\lambda), (\mathbf{c}_\lambda)$ are bases of \mathbf{A} . Let $(\mathcal{R}(\mathfrak{g}), \otimes)$ be the tensor category of finite dimensional \mathfrak{g} -modules. We have proved the following theorem.

Theorem. *The tensor categories $(\mathcal{P}^\circ, \odot), (\mathcal{R}(\mathfrak{g}), \otimes)$ are equivalent. The group homomorphism such that $\mathbf{b}_\lambda \mapsto V(\lambda)$ is a ring isomorphism $\mathbf{A} \xrightarrow{\sim} \mathbf{R}(\mathfrak{g})$. Moreover, we have $\mathbf{b}_\lambda = \sum_{\mu} \dim V(\lambda)_\mu \cdot \mathbf{c}_\mu$.*

5.2. In this subsection we consider the non simply laced case. Our construction is based on [L1, §11]. Assume that $\underline{\mathfrak{g}}$ is a non simply laced, simple, complex Lie algebra. Fix a simply laced simple Lie algebra \mathfrak{g} and a diagram automorphism a of \mathfrak{g} such that the Dynkin graph of $\underline{\mathfrak{g}}$ is deduced from the Dynkin graph of \mathfrak{g} as in [L1, §14]. Let n be the order of the automorphism a ($n = 2$ for types B_k, C_k, F_4 , and $n = 3$ for type G_2). The automorphism a is identified with a permutation of the set $I \times H$, see §3.1, such that

$$a(h') = a(h)', \quad a(h'') = a(h''), \quad a(\bar{h}) = \overline{a(h)}.$$

Let $\langle a \rangle$ be the cyclic group of automorphisms of (I, H) generated by a . Let \underline{I} be the set of $\langle a \rangle$ -orbits in I , and let $\underline{P}^+ = (P^+)^a, \underline{Q}^+ = (Q^+)^a$ be the corresponding sub-semigroups of P^+, Q^+ . The simple root $\alpha_{\underline{i}}$ and the fundamental weight $\omega_{\underline{i}}$ of $\underline{\mathfrak{g}}$ are identified with the sums $\sum_{i \in \underline{i}} \alpha_i \in \underline{Q}^+, \sum_{i \in \underline{i}} \omega_i \in \underline{P}^+$. For any $\lambda \in P^+, \alpha \in$

Q^+ , the diagram automorphism induces natural isomorphisms $Q_{\lambda\alpha} \xrightarrow{\sim} Q_{a(\lambda),a(\alpha)}$, $N_{\lambda\alpha} \xrightarrow{\sim} N_{a(\lambda),a(\alpha)}$. Let denote them by a again.

To avoid confusions, finite dimensional representations of \mathfrak{g} , $\underline{\mathfrak{g}}$ are denoted by $V(\lambda)$, $\underline{V}(\lambda)$ respectively. The subsets of Q^+ , \underline{Q}^+ defined in §2.1 are denoted by $\Lambda(\lambda)$, $\Lambda^+(\lambda)$ and $\underline{\Lambda}(\lambda)$, $\underline{\Lambda}^+(\lambda)$ respectively.

Fix $\lambda, \lambda' \in \underline{P}^+$ and $\alpha \in Q^+$. Following [L1, §11] we consider new categories ${}^a\mathcal{P}_\lambda$, ${}^a\mathcal{P}_{\lambda\lambda'}$. An object of ${}^a\mathcal{P}_\lambda$ is a pair (P, θ) , where $P \in \text{Ob}(\mathcal{P}_\lambda)$ and $\theta : a^*P \xrightarrow{\sim} P$ is an isomorphism such that the composition

$$a^{*n}P \longrightarrow \cdots \longrightarrow a^{*2}P \xrightarrow{a^*\theta} a^*P \xrightarrow{\theta} P$$

is the identity. A morphism $(P, \theta) \rightarrow (P', \theta')$ is a morphism $f : P \rightarrow P'$ such that $f\theta = \theta'(a^*f)$. The category ${}^a\mathcal{P}_{\lambda\lambda'}$ is constructed in the same way. Both categories are Abelian. For any functor $F : \mathcal{P}_\lambda \rightarrow \mathcal{P}_{\lambda'}$ and for any isomorphism of functor $a^*F \xrightarrow{\sim} Fa^*$ there is the functor ${}^aF : {}^a\mathcal{P}_\lambda \rightarrow {}^a\mathcal{P}_{\lambda'}$ such that ${}^aF(P, \theta) = (F(P), \theta^F)$ where θ^F is the composition of the chain of maps

$$a^*F(P) \longrightarrow F(a^*P) \xrightarrow{F(\theta)} F(P).$$

The functor a^* on \mathcal{P}_λ has the order n , where $n = 2$ or 3 . Let ${}^a\mathcal{I}_\lambda$ be the full subcategory of ${}^a\mathcal{P}_\lambda$ whose objects are the pairs (P, θ) such that $P \simeq P' \oplus a^*P' \oplus \cdots \oplus (a^*)^{n-1}P'$ for some $P' \in \mathcal{P}_\lambda$, and θ is an isomorphism carrying the direct summand $(a^*)^jP' \subset a^*P$ onto the direct summand $(a^*)^jP' \subset P$. The objects of ${}^a\mathcal{I}_\lambda$ are said to be traceless.

The automorphism a preserves the stratification of N_λ . Since $IC_{\lambda\alpha}$ is canonically attached to $N_{\lambda\alpha}^\heartsuit$, there is a canonical isomorphism $a^*IC_{\lambda,a(\alpha)} \xrightarrow{\sim} IC_{\lambda\alpha}$. If $\alpha \in \underline{Q}^+$ the corresponding object in ${}^a\mathcal{P}_\lambda$ is denoted by ${}^aIC_{\lambda\alpha}$. Let $\mu_n \subset \mathbb{C}^\times$ be the set of n -th roots of unity. For any $\zeta \in \mu_n$ and any $Q = (P, \theta) \in \text{Ob}({}^a\mathcal{P}_\lambda)$ we put $Q(\zeta) = (P, \zeta\theta)$. If $\alpha \notin \underline{Q}^+$ and $\zeta_1, \dots, \zeta_n \in \mu_n$, let ${}^aIC_{\lambda\alpha}(\zeta_1, \dots, \zeta_n)$ be the object of ${}^a\mathcal{P}_\lambda$ associated to the perverse sheaf

$$P = IC_{\lambda\alpha} \oplus IC_{\lambda,a(\alpha)} \oplus \cdots \oplus IC_{\lambda,a^{n-1}(\alpha)}$$

and the isomorphism $a^*P \xrightarrow{\sim} P$ which maps the summand $a^*IC_{\lambda,a^i(\alpha)}$ onto the summand $IC_{\lambda,a^{i-1}(\alpha)}$ by ζ_{i+1} times the canonical isomorphism. A simple object in ${}^a\mathcal{P}_\lambda$ is isomorphic either to ${}^aIC_{\lambda\alpha}(\zeta)$ for some $\alpha \in \underline{Q}^+$ and $\zeta \in \mu_n$, or to ${}^aIC_{\lambda\alpha}(\zeta_1, \dots, \zeta_n)$ for some $\alpha \in Q^+ \setminus \underline{Q}^+$ and $\zeta_1, \dots, \zeta_n \in \mu_n$. Let ${}^1\mathcal{P}_\lambda$ be the full subcategory of ${}^a\mathcal{P}_\lambda$ whose objects are isomorphic to finite direct sums of the objects ${}^aIC_{\lambda\alpha}$.

The image by the functor $\pi_{\lambda\alpha}!$ of the obvious isomorphism $a^*\mathbb{C}_{Q_{\lambda,a(\alpha)}} \xrightarrow{\sim} \mathbb{C}_{Q_{\lambda\alpha}}$ is an isomorphism $a^*L_{\lambda,a(\alpha)} \xrightarrow{\sim} L_{\lambda\alpha}$. If $\alpha \in \underline{Q}^+$ the corresponding object in ${}^a\mathcal{P}_\lambda$ is denoted by ${}^aL_{\lambda\alpha}$. Assume that $\beta \in Q^+$ is such that $\alpha \geq \beta$ and $N_{\lambda\beta}^\heartsuit \neq \emptyset$. Fix an element $x_\beta \in N_{\lambda\beta}^\heartsuit$. One proves as in [N2, Theorem 3.3.2] that there are $\langle a \rangle$ -invariant open sets

$$U_\alpha \subset \langle a \rangle(N_{\lambda\alpha}), \quad U_\beta^\heartsuit \subset \langle a \rangle(N_{\lambda\beta}^\heartsuit), \quad U_{\alpha-\beta} \subset \langle a \rangle(N_{\lambda-\beta, \alpha-\beta})$$

containing x_β , x_β , 0 respectively, and a commutative square

$$\begin{array}{ccc} U_\alpha & \xrightarrow{\sim} & U_\beta^\heartsuit \times U_{\alpha-\beta} \\ \pi \uparrow & & \uparrow \text{Id} \times \pi \\ \pi^{-1}(U_\alpha) & \xrightarrow{\sim} & U_\beta^\heartsuit \times \pi^{-1}(U_{\alpha-\beta}), \end{array}$$

where π denotes either $\pi_{\lambda\alpha}$ or $\pi_{\lambda-\beta, \alpha-\beta}$. The horizontal maps are analytic $\langle a \rangle$ -equivariant isomorphisms carrying the element $x_\beta \in U_\alpha$ to $(x_\beta, 0) \in U_\beta^\heartsuit \times U_{\alpha-\beta}$. By (5.1.2) we have

$$L_{\lambda\alpha} \simeq \bigoplus_{\beta \in Q^+} H_{\text{top}}(F_{\lambda-\beta, \alpha-\beta}) \otimes IC_{\lambda\beta},$$

and the isomorphism $a^*L_{\lambda, a(\alpha)} \xrightarrow{\sim} L_{\lambda\alpha}$ maps the direct summand

$$H_{\text{top}}(a(F_{\lambda-\beta, \alpha-\beta})) \otimes a^*IC_{\lambda, a(\beta)} \quad \text{onto} \quad H_{\text{top}}(F_{\lambda-\beta, \alpha-\beta}) \otimes IC_{\lambda\beta}$$

in the obvious way. By [X, Theorem 3.2.1], if $\alpha, \beta \in Q^+$ the number of irreducible components of $F_{\lambda-\beta, \alpha-\beta}$ which are mapped to themselves by a is the multiplicity $\dim \underline{V}(\lambda - \beta)_{\lambda-\alpha}$. Thus ${}^aL_{\lambda\alpha} = {}^1L_{\lambda\alpha} \oplus I_{\lambda\alpha}$ where

$$(1) \quad {}^1L_{\lambda\alpha} \simeq \bigoplus_{\beta \in Q^+} \underline{V}(\lambda - \beta)_{\lambda-\alpha} \otimes {}^aIC_{\lambda\beta} \in \text{Ob}({}^1\mathcal{P}_\lambda), \quad I_{\lambda\alpha} \in \text{Ob}({}^a\mathcal{I}_\lambda).$$

Assume that $\lambda = \lambda' + \lambda''$ in \underline{P}^+ . The maps $\iota_\pm, \kappa_\pm, \phi$ commute to the automorphism a of N_λ . Thus, there is a natural isomorphism $a^*\text{res}_{\lambda'\lambda''} \xrightarrow{\sim} \text{res}_{\lambda'\lambda''} a^*$. We get the functor ${}^a\text{res}_{\lambda'\lambda''} : {}^a\mathcal{P}_\lambda \rightarrow {}^a\mathcal{P}_{\lambda'\lambda''}$. Lemma 5.1.1 implies the following.

Lemma. *For any $\alpha \in Q^+$ there are traceless objects I, I' such that*

$$\begin{aligned} (a) \quad & {}^a\text{res}_{\lambda'\lambda''}({}^aL_{\lambda\alpha}) = I \oplus \bigoplus_{\alpha=\alpha'+\alpha''} {}^aL_{\lambda'\alpha'} \boxtimes {}^aL_{\lambda''\alpha''}, \\ (b) \quad & {}^a\text{res}_{\lambda'\lambda''}({}^aIC_{\lambda\alpha}) = I' \oplus \bigoplus_{\alpha', \alpha''} \underline{V}(\lambda' - \alpha', \lambda'' - \alpha'')_{\lambda-\alpha} \otimes ({}^aIC_{\lambda'\alpha'} \boxtimes {}^aIC_{\lambda''\alpha''}). \end{aligned}$$

For any $\beta \in \bigwedge^+(\lambda) \cap Q^+$, there is an obvious isomorphism of functors $a^*p_{\lambda\beta} \xrightarrow{\sim} p_{\lambda\beta} a^*$. The corresponding functor ${}^a p_{\lambda\beta} : {}^a\mathcal{P}_\lambda \rightarrow {}^a\mathcal{P}_{\lambda-\beta}$ is exact and satisfies

$${}^a p_{\lambda\beta}({}^aIC_{\lambda\alpha}) = \begin{cases} {}^aIC_{\lambda-\beta, \alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else.} \end{cases}$$

We have also, see (5.2.1),

$${}^a p_{\lambda\beta}({}^aL_{\lambda\alpha}) \simeq \begin{cases} {}^aL_{\lambda-\beta, \alpha-\beta} & \text{if } \alpha \geq \beta \\ 0 & \text{else.} \end{cases}$$

Let $\mathbf{K}({}^a\mathcal{P}_\lambda)$ be the Grothendieck group of ${}^a\mathcal{P}_\lambda$. The class of an object P is still denoted by P . Let $\mathbf{k} \subset \mathbb{C}$ be subring generated by μ_n . Let \mathcal{K}'_λ be the quotient of $\mathbf{K}({}^a\mathcal{P}_\lambda) \otimes \mathbf{k}$ by the relations :

- $P(\zeta) = P \otimes \zeta$ for any $\zeta \in \mu_n$,
- the class of a traceless object is zero.

Let $\mathcal{K}_\lambda \subset \mathcal{K}'_\lambda$ be the subgroup spanned by the classes of objects in ${}^1\mathcal{P}_\lambda$, and let $\mathbf{A}_\lambda = \mathcal{K}_\lambda^*$ be the dual group. Using the maps ${}^a p_{\lambda\beta}$ we construct, as in §4.3, an inductive system of groups (\mathbf{A}_λ) . The limit, denoted by \mathbf{A} , is endowed with a product $\mathbf{A} \otimes \mathbf{A} \rightarrow \mathbf{A}$, and two distinguished bases (\mathbf{b}_λ) , (\mathbf{c}_λ) associated to the families $({}^aIC_{\lambda+\beta, \beta})$, $({}^aL_{\lambda+\beta, \beta})$.

Theorem. *The group homomorphism such that $\mathbf{b}_\lambda \mapsto \underline{V}(\lambda)$ is a ring isomorphism $\mathbf{A} \xrightarrow{\sim} \mathbf{R}(\underline{\mathfrak{g}})$. Moreover, we have $\mathbf{b}_\lambda = \sum_\mu \dim \underline{V}(\lambda)_\mu \cdot \mathbf{c}_\mu$.*

5.3. In this subsection we explain how a similar construction gives a natural restriction map $\mathbf{GR} \rightarrow \mathbf{R}(\underline{\mathfrak{g}}) \otimes \mathbb{A}$. Consider the diagram

$$\begin{array}{ccccc} Q_\lambda & \xleftrightarrow{\tilde{\iota}^\pm} & Q_\lambda^{\pm\gamma} & \xrightarrow{\tilde{\kappa}^\pm} & Q_\gamma \\ \downarrow & \square & \downarrow & & \downarrow \\ N_\lambda & \xleftrightarrow{\iota^\pm} & N_\lambda^{\pm\gamma} & \xrightarrow{\kappa^\pm} & N_\gamma. \end{array}$$

Set $\varepsilon_\gamma = \varepsilon_{\gamma\gamma}$. Let κ_η^\pm be the relative dimension of $\tilde{\kappa}_\pm$ above the component $Q_{\gamma\eta}$. The same computations as in Lemma 3.5.2 or in (5.1.1) give

$$\begin{aligned} \kappa_\eta^- &= d_{\lambda\alpha}/2 - d_{\gamma\eta}, & \kappa_\eta^+ &= d_{\lambda\alpha}/2, \\ \varepsilon_\gamma - \varepsilon_{\gamma-\delta} &= d_{\gamma\delta}, & d_{\gamma\eta} - d_{\gamma-\delta, \eta-\delta} &= d_{\gamma\delta}, \end{aligned}$$

for any $\delta \in \bigwedge^+(\gamma)$, $\eta \succeq \delta$. Consider the functor

$$\text{res}_\gamma^\pm = \kappa_{\pm!} \iota^* : \mathcal{D}(N_\lambda)^\heartsuit \rightarrow \mathcal{D}(N_\gamma)^\heartsuit.$$

By base change we get, for any $\eta \in \bigwedge^+(\gamma)$,

$$\text{res}_\gamma^\pm L_{\lambda\alpha} = \pi_{\gamma!} \tilde{\kappa}_{\pm!} \tilde{\iota}_\pm^* \mathbb{C}_{Q_{\lambda\alpha}}[d_{\lambda\alpha}] = \bigoplus_\eta \pi_{\gamma!} \mathbb{C}_{Q_{\gamma\eta}}[d_{\lambda\alpha} - 2\kappa_\eta^\pm] = \bigoplus_\eta L_{\gamma\eta}[\mp d_{\gamma\eta}].$$

Lemma. (a) *For any complex $P \in \text{Ob}(\mathcal{P}_\lambda)$ the complex $\text{res}_\gamma^\pm(P)$ belongs to $\text{Ob}(\mathcal{P}_\gamma)$.*
 (b) *We have $\mathbb{D} \circ \text{res}_\gamma^+ = \text{res}_\gamma^- \circ \mathbb{D}$.*

The corresponding group homomorphism $v^{\varepsilon_\gamma} \text{res}_\gamma^+ : \mathbf{K}(\mathcal{P}_\lambda) \rightarrow \mathcal{K}_\gamma$ is compatible with the projective systems in §5.1, §4.2. Let $\text{res} : \mathbf{GA} \rightarrow \mathbf{A} \otimes \mathbb{A}$ be the inductive limit of the system of maps dual to $v^{\varepsilon_\gamma} \text{res}_\gamma^+$.

Proposition. *The element $\text{res}(\mathbf{b}_\gamma)$ belongs to $\bigoplus_\lambda \mathbb{N}[v^{-1}, v] \cdot \mathbf{b}_\lambda$ for all $\gamma \in X^+$. If $\gamma \in G_\lambda^{\vee, \text{ad}}$ then $\text{res}(\mathbf{c}_\gamma) = v^{\varepsilon_\gamma} \mathbf{c}_\lambda$.*

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Michela Varagnolo
Département de Mathématiques
Université de Cergy-Pontoise
2 Av. A. Chauvin
95302 Cergy-Pontoise Cedex
France
email: michela.varagnolo@u-cergy.fr

Eric Vasserot
Département de Mathématiques
Université de Cergy-Pontoise
2 Av. A. Chauvin
95302 Cergy-Pontoise Cedex
France
email: eric.vasserot@u-cergy.fr